JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 30, No. 2, March–April 2007

Stable Adaptation in the Presence of Actuator Constraints with Flight Control Applications

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DOI: 10.2514/1.26984

For a class of multi-input dynamical systems with unknown parameters and matched uncertainties, a direct model reference adaptive control framework is developed that provides stable adaptation in the presence of multi-input constraints. The design is Lyapunov based and ensures global asymptotic tracking for open-loop stable systems. For unstable systems an estimate for the domain of attraction is derived based on the saturation magnitudes of actuators and system parameters. The theoretical results are verified using linear roll/yaw dynamics of an F-16 aircraft, in which benefits of the developed methodology are demonstrated in the presence of aerodynamic uncertainties and control failures.

I. Introduction

IN THIS paper, we present a direct model reference adaptive control design method for multi-input systems in the presence of control magnitude constraints. The design of such methods has attracted a large amount of research effort over the past years [1–8]. The first attempt to modify an adaptive control scheme in the presence of saturation was proposed by Monopoli in [9] but without any formal proof of stability. In [3], a formal proof was given for the same idea, but it was limited to single-input systems. In [10], the results of [3] were extended to the class of multi-input systems, assuming an ellipsoid-based saturation function, as it is shown in Fig. 1. Because the actuator limits were artificially reduced, this type of approach introduced extra conservatism into the design.

Here, we extend the result reported in [10] and prove closed-loop system stability in the presence of rectangular saturation function sat(u), which corresponds to the actual saturation bounds, Fig. 1. Similar to [10], we show that for open-loop stable (unstable) systems global (local) stability results can be attained. We also derive an upper bound for the corresponding closed-loop system stability domain and prove that it depends upon the system parameters and the actuator limits. In addition, we extend the "μ-mod" architecture that was introduced in [11] for single-input systems to multi-input systems.

This paper is organized as follows. Mathematical preliminaries are given in Sec. II. In Sec. III we formulate the design problem for linear in parameters adaptive control with input saturation. Section IV defines the proposed μ-modification to the adaptive signal and discusses some of its properties. In Sec. V, reference model dynamics and the classical matching conditions are formulated. Stability properties of the μ-mod based adaptive control are analyzed in Sec. VI. Section VII discusses the implementation of the method for augmenting a baseline nominal linear controller. In Sec. VIII, flight control design examples are given that demonstrate the benefits of the method. Conclusions, recommendations, and future research directions are given at the end of the paper. Unless otherwise mentioned, $\|\cdot\|$ denotes the 2-norm of a vector.

II. Mathematical Preliminaries on Multi-Input Control Saturation

Consider an m-dimensional vector $y \in \mathbb{R}^m$, and introduce

$$\text{sat}\,(y) = \begin{pmatrix} \text{sat}(y_1) \\ \vdots \\ \text{sat}(y_m) \end{pmatrix} \tag{1}$$

where sat(y_i) is defined as

$$\text{sat}\,(y_i) = \begin{cases} y_i, & |y_i| \le 1 \\ \text{sgn}(y_i), & |y_i| > 1 \end{cases} \tag{2}$$

Recall that the ∞-norm of a vector is

$$\|y\|_\infty = \max_i |y_i|$$

Assuming that $y \neq 0$, scale the vector y by its ∞-norm and let

$$y_\perp = \frac{y}{\|y\|_\infty}$$

Geometrically speaking, $y_\perp$ is the projection of y onto a unit cube, where the latter is understood in terms of the ∞-norm. Figure 2 shows that the projection $y_\perp$ touches one of the sides of the unit cube $\|y\|_\infty \le 1$, which, in turn, defines the saturation limits for each of the components of the vector y. Notice that $y_\perp$ is a direction preserving scaled version of the original vector y and it does not violate the saturation constraints. However, the saturation function sat(y) is different from the projection vector $y_\perp$. In fact, from the definitions in Eqs. (1) and (2) it follows that

$$\text{sat}\,(y) = \begin{cases} y, & \|y\|_\infty \le 1 \\ y_\perp + \bar{y}, & \|y\|_\infty > 1 \end{cases} \tag{3}$$

where the components of the newly introduced vector $\bar{y}$ are

$$\bar{y}_i = \text{sat}(y_i) - y_{\perp_i} = \begin{cases} y_i - y_{\perp_i}, & |y_i| \le 1 \\ \text{sgn}(y_i) - y_{\perp_i}, & |y_i| > 1 \end{cases}$$

This vector originates from the maximum achievable vector and points in the direction of y to the actually achieved (under the sat-function constraints) vector. Further notice that

Presented as Paper 6342 at the AIAA Guidance, Navigation, and Control Conference and Exhibit, Keystone, Colorado, 21–24 August 2006; received 2 August 2006; revision received 2 October 2006; accepted for publication 4 October 2006. Copyright © 2006 by Eugene Lavretsky and Naira Hovakimyan. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the $10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0731-5090/07 $10.00 in correspondence with the CCC.

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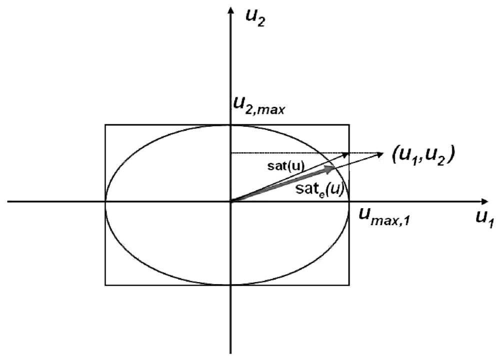

Fig. 1 Elliptical saturation function.

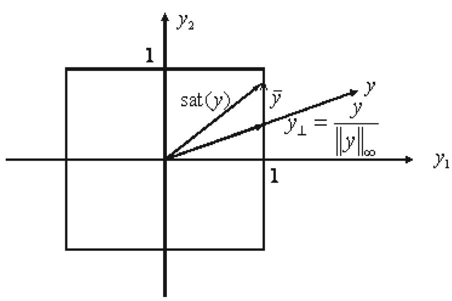

Fig. 2 Geometric illustration of vectors and norms.

$$\bar{y}_i = \begin{cases} \left(1 - \frac{1}{\|y\|_\infty}\right) y_i, & |y_i| \le 1 \\ (1 - |y_{\perp_i}|)\operatorname{sgn}(y_i), & |y_i| > 1 \end{cases} \tag{4}$$

Therefore

$$|\bar{y}_i| = \begin{cases} \left|1 - \frac{1}{\|y\|_\infty}\right| |y_i|, & |y_i| \le 1 \\ \left|1 - \frac{|y_i|}{\|y\|_\infty}\right|, & |y_i| > 1 \end{cases} \tag{5}$$

It immediately follows that

$$|\bar{y}_i| \le \left|1 - \frac{1}{\|y\|_\infty}\right| \tag{6}$$

If $\|y\|_\infty > 1$, then it follows from Eq. (6) that $|\bar{y}_i| < 1$ for all $i = 1, \ldots, m$. Hence, $\|\bar{y}\|_\infty < 1$ or

$$\|y\|_\infty > 1 \Rightarrow \|\bar{y}\| < 1 \Rightarrow \|\bar{y}\|_\infty < \|y\|_\infty \tag{7}$$

One has the obvious relationship between the norms of a vector

$$\|y\|_\infty \le \|y\| \le \sqrt{m}\|y\|_\infty$$

or equivalently

$$\frac{\|y\|}{\sqrt{m}} \le \|y\|_\infty \le \|y\| \tag{8}$$

Using Eq. (8) in Eq. (7) for $\|y\|_\infty > 1$, we have

$$\|\bar{y}\| < \sqrt{m}\|\bar{y}\|_\infty < \sqrt{m}\|y\|_\infty < \sqrt{m}\|y\| \tag{9}$$

Lemma 1: If $\|y\|_\infty > 1$, then the following inequalities are true:

$$0 < \|\mathrm{sat}(y)\| - \|\bar{y}\| \le \sqrt{m} \tag{10}$$

Proof: Indeed,

$$\begin{aligned} |\,\|\mathrm{sat}(y)\| - \|\bar{y}\|\,| &\le \|\mathrm{sat}(y) - \bar{y}\| = \|y_\perp\|_2 \\ &= \frac{\|y\|_2}{\|y\|_\infty} \le \frac{\sqrt{m}\|y\|_\infty}{\|y\|_\infty} = \sqrt{m} \end{aligned}$$

On the other hand, to prove $0 < \|\mathrm{sat}(y)\| - \|\bar{y}\|_2$, it is sufficient to prove that

$$\begin{aligned} (\|\mathrm{sat}(y)\| - \|\bar{y}\|)(\|\mathrm{sat}(y)\| + \|\bar{y}\|) &= \|\mathrm{sat}(y)\|_2^2 - \|\bar{y}\|_2^2 \\ &= \sum_{i=1}^{m}((\mathrm{sat}(y_i))^2 - (\bar{y}_i)^2) > 0 \end{aligned} \tag{11}$$

To this end, notice that from Eqs. (2) and (4) it follows that

$$(\mathrm{sat}(y_i))^2 - (\bar{y}_i)^2 = \begin{cases} y_i^2 - \left(1 - \frac{1}{\|y\|_\infty}\right)^2 y_i^2, & |y_i| \le 1 \\ 1 - \left(1 - \frac{|y_i|}{\|y\|_\infty}\right)^2, & |y_i| > 1 \end{cases}$$

If $\|y\|_\infty > 1$, then

$$0 < \left(1 - \frac{1}{\|y\|_\infty}\right)^2 < 1$$

At the same time, from the definition of $\|y\|_\infty$ it follows that

$$0 \le \left(1 - \frac{|y_i|}{\|y\|_\infty}\right)^2 \le 1$$

Because $\|y\|_\infty > 1$, then $y \ne 0$, and therefore at least one i exists for which

$$0 \le \left(1 - \frac{|y_i|}{\|y\|_\infty}\right)^2 < 1$$

Hence

$$\sum_{i=1}^{m}((\mathrm{sat}(y_i))^2 - (\bar{y}_i)^2) > 0$$

Thus, the lower bound in Eq. (11) holds and

$$\|\mathrm{sat}(y)\| - \|\bar{y}\| > 0$$

which leads to Eq. (10). The proof is complete.

Introduce

$$X_{\max} = \begin{bmatrix} X_{\max_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{\max_m} \end{bmatrix}$$

where $X_{\max_i} \ne 0$, and define

$$y = X_{\max}^{-1} x$$

Then it is easy to see that the projection of y onto a unit cube (defined by the ∞-norm) is equivalent to projecting vector x onto an m-dimensional rectangle with each side bounded by $X_{\max_i}$.

III. Problem Formulation

Let the system dynamics propagate according to the following differential equation:

$$\dot{x}(t) = Ax(t) + B\Lambda u(t) \tag{12}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the control input, A is an unknown $(n \times n)$ matrix, B is a known $(n \times m)$ constant matrix, and Λ is an unknown constant diagonal $(m \times m)$ matrix with positive diagonal elements. The control input $u \in \mathbb{R}^m$ is amplitude limited and is calculated using the following *static actuator model*:

$$u(t) = \begin{pmatrix} u_{\max_1} \operatorname{sat}\left(\frac{u_{c_1}(t)}{u_{\max_1}}\right) \\ \vdots \\ u_{\max_m} \operatorname{sat}\left(\frac{u_{c_m}(t)}{u_{\max_m}}\right) \end{pmatrix} \tag{13}$$

Here $u_{c_1}(t), \ldots, u_{c_m}(t)$ are the components of the commanded control vector $u_c(t)$, while $u_{\max_1}, \ldots, u_{\max_m}$ are the actuator saturation limits. Equivalently, we can rewrite this as

$$u(t) = U_{\max} \operatorname{sat}(U_{\max}^{-1} u_c(t)) \tag{14}$$

where

$$U_{\max} = \begin{bmatrix} u_{\max_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_{\max_m} \end{bmatrix}$$

Thus, for $i = 1, \ldots, m$, componentwise one gets

$$u_i(t) = u_{\max_i} \operatorname{sat}\left(\frac{u_{c_i}(t)}{u_{\max_i}}\right) = \begin{cases} u_{c_i}(t), & |u_{c_i}(t)| \le u_{\max_i} \\ u_{\max_i} \operatorname{sgn}(u_{c_i}(t)), & |u_{c_i}(t)| > u_{\max_i} \end{cases} \tag{15}$$

Lemma 2: For any $u_c(t) \in \mathbb{R}^m$ there exists a bounded vector $\bar{u}(t) \in \mathbb{R}^m$ such that $\forall\ t > 0$ and the output of the static actuator model (14) and (15) can be written as

$$u(t) = \begin{cases} u_c(t), & \|U_{\max}^{-1} u_c(t)\|_\infty \le 1 \\ u_{c_\perp}(t) + \bar{u}(t), & \|U_{\max}^{-1} u_c(t)\|_\infty > 1 \end{cases} \tag{16}$$

where

$$u_{c_\perp}(t) = \frac{u_c(t)}{\|U_{\max}^{-1} u_c(t)\|_\infty}$$

and the components of $\bar{u}(t)$ are

$$\bar{u}_i(t) = \begin{cases} u_{c_i}(t) - u_{c_{\perp i}}(t), & |u_{c_i}(t)| \le u_{\max_i} \\ \operatorname{sgn}(u_{c_i}(t)) - u_{c_{\perp i}}(t), & |u_{c_i}(t)| > u_{\max_i} \end{cases}$$

The proof follows from definitions in Eqs. (14) and (15). Moreover, from Lemma 2 and the relationships in Eq. (9) it immediately follows that

$$\|\bar{u}(t)\| < \|u(t)\| \le \sqrt{m} u_{\max} \tag{17}$$

where $u_{\max} = \max\{u_{\max_1}, \ldots, u_{\max_m}\}$.

Rewrite the system dynamics in Eq. (12) in the form:

$$\dot{x}(t) = Ax(t) + B\Lambda(u_c(t) + \Delta u(t)) \tag{18}$$

where $\Delta u(t) = u(t) - u_c(t)$ denotes the *control deficiency* due to the actuators saturation limits:

$$\Delta u(t) = [\Delta u_1(t) \cdots \Delta u_m(t)]^\top \tag{19}$$

$$\Delta u_i(t) = \begin{cases} 0, & |u_{c_i}(t)| \le u_{\max_i} \\ (u_{\max_i} - |u_{c_i}(t)|)\operatorname{sgn}(u_{c_i}(t)), & |u_{c_i}(t)| > u_{\max_i} \end{cases} \tag{20}$$

Notice that from Eq. (16) it follows that

$$\Delta u(t) = \begin{cases} 0, & \|U_{\max}^{-1} u_c(t)\|_\infty \le 1 \\ u_{c_\perp}(t) + \bar{u}(t) - u_c(t), & \|U_{\max}^{-1} u_c(t)\|_\infty > 1 \end{cases} \tag{21}$$

Consider the so-called ideal reference model dynamics, driven by a uniformly bounded reference input $\{r \in \mathbb{R}^m : \|r(t)\| \le r_{\max}\}$:

$$\dot{x}_m^*(t) = A_m x_m^*(t) + B_m r(t) \tag{22}$$

In Eq. (22), $x_m^* \in \mathbb{R}^n$ is the state of the reference model, A_m is a Hurwitz matrix, and the pair (A_m, B_m) is controllable. The control design problem, addressed in this paper, can be stated as follows:

Given ideal reference model (22), define an adaptive control command $u_c(t)$ and, if necessary, augment the input $r(t)$ to the reference model, so that the state $x(t)$ of the system (12) in the presence of multi-input constraint (14) tracks the state $x_m(t)$ of the augmented reference model asymptotically, while all the signals in both systems remain bounded.

IV. Positive μ-Modification and Closed-Loop System Dynamics

The main challenge in the design of an adaptive controller for the system in Eqs. (12) and (14) is associated with the control deficiency $\Delta u(t) = u(t) - u_c(t)$ that appears in Eq. (18). Using this signal, in [10] a modification to the reference model dynamics was suggested, assuming that $u(t)$ always lies inside the ellipse, Fig. 1. This assumption streamlined the stability proof in [10]. Obviously, the elliptical bound in Fig. 1 is conservative as compared to the true saturation bounds in Fig. 1. In this paper, with the help of the preliminary definitions and lemmas introduced in Sec. II, we extend the proof from [10] to the true saturation function, thus enabling the control system to have better tracking performance. In addition, we introduce the μ-modification from [11] to enable prevention of control saturation if needed. Rationale for using the μ-mod in real-life control applications was provided in [11]. We note that the case $\mu = 0$ corresponds to the true saturation bound in Fig. 1.

Choose a nonnegative constant $0 \le \delta < 1$, and introduce the so-called virtual control bounds $u_{\max_i}^\delta = (1 - \delta) u_{\max_i}$ for every $i = 1, \ldots, m$. Direct adaptive model reference control architecture with μ-modification is defined as

$$u_c(t) = u_{\mathrm{ad}}(t) + \mu \Delta u_c(t) \tag{23}$$

$$u_{\mathrm{ad}}(t) = K_x^\top(t) x(t) + K_r^\top(t) r(t) \tag{24}$$

where $\Delta u_c(t) = [\Delta u_{c_1}(t) \cdots \Delta u_{c_m}(t)]^\top$, and

$$\Delta u_{c_i}(t) = u_{\max_i}^\delta \operatorname{sat}\left(\frac{u_{c_i}(t)}{u_{\max_i}^\delta}\right) - u_{c_i}(t), \qquad i = 1, \ldots, m \tag{25}$$

The main purpose of introducing δ and μ is to define a "safety zone" to the actuator boundaries and to modify the commanded control when it enters the zone, correspondingly. As discussed later in Corollary 1, this modification allows one to prevent the actuator saturation phenomenon at all times.

Notice that the relation in Eq. (23) defines commanded control input $u_c(t)$ implicitly. The next lemma formulates a sufficient condition for the existence of its explicit solution.

Lemma 3: Let $\tilde{U}_{\max} = (1 - \delta) U_{\max}$. If $\mu \ge 0$, then the explicit solution to Eq. (23) is given by a convex combination of $u_{\mathrm{ad}}(t)$ and $u_{\max}^\delta \operatorname{sat}[u_{\mathrm{ad}}(t)/u_{\max}^\delta]$ $\forall\ \delta \ge 0$ and $\forall\ t > 0$:

$$u_c(t) = \frac{1}{1 + \mu}\left(u_{\mathrm{ad}}(t) + \mu \tilde{U}_{\max} \operatorname{sat}(\tilde{U}_{\max}^{-1} u_{\mathrm{ad}}(t))\right) \tag{26}$$

Moreover, a componentwise representation of Eq. (26) is given by

$$u_{c_i}(t) = \begin{cases} u_{\mathrm{ad}_i}(t), & |u_{\mathrm{ad}_i}(t)| \le u_{\max_i}^\delta \\ \frac{1}{1+\mu}(u_{\mathrm{ad}_i}(t) + \mu u_{\max_i}^\delta), & u_{\mathrm{ad}_i}(t) > u_{\max_i}^\delta \\ \frac{1}{1+\mu}(u_{\mathrm{ad}_i}(t) - \mu u_{\max_i}^\delta), & u_{\mathrm{ad}_i}(t) < -u_{\max_i}^\delta \end{cases} \tag{27}$$

Proof: If $|u_{c_i}(t)| \le u_{\max_i}^\delta$, then $\Delta u_{c_i}(t) = 0$, and the first relationship in Eq. (27) immediately takes place. If $|u_{c_i}(t)| > u_{\max_i}^\delta$, then using Eqs. (14) and (23), we get

$$u_{c_i}(t) = u_{\mathrm{ad}_i}(t) + \mu(u_{\max_i}^\delta \operatorname{sgn}(u_{c_i}(t)) - u_{c_i}(t)) \tag{28}$$

or equivalently

$$
\begin{aligned}
u_{c_i}(t) &= \frac{1}{1+\mu}\left(u_{\mathrm{ad}_i}(t) + \mu u^{\delta}_{\max_i}\operatorname{sgn}(u_{c_i}(t))\right) \\
&= \begin{cases} \frac{1}{1+\mu}\left(u_{\mathrm{ad}_i}(t) + \mu u^{\delta}_{\max_i}\right), & u_{c_i} > u^{\delta}_{\max_i} \\ \frac{1}{1+\mu}\left(u_{\mathrm{ad}_i}(t) - \mu u^{\delta}_{\max_i}\right), & u_{c_i} < -u^{\delta}_{\max_i} \end{cases}
\end{aligned} \tag{29}
$$

It is easy to see that since $\mu \geq 0$, then Eq. (29) is equivalent to Eq. (27). The proof is complete.

Remark 1: Solving Eq. (23) for $\Delta u_{c_i}(t)$ and substituting $u_{c_i}(t)$ from Eq. (27), one obtains

$$
\begin{aligned}
\Delta u_{c_i}(t) &= \frac{1}{\mu}(u_{c_i}(t) - u_{\mathrm{ad}_i}(t)) = \frac{1}{\mu}\left(\frac{1}{1+\mu}\left(u_{\mathrm{ad}_i}(t) \right.\right.\\
&\quad \left.\left. + \mu u^{\delta}_{\max_i}\operatorname{sat}\left(\frac{u_{\mathrm{ad}_i}(t)}{u^{\delta}_{\max}}\right)\right) - u_{\mathrm{ad}_i}(t)\right) \\
&= \frac{1}{1+\mu}\left(u^{\delta}_{\max}\operatorname{sat}\left(\frac{u_{\mathrm{ad}_i}(t)}{u^{\delta}_{\max}}\right) - u_{\mathrm{ad}_i}(t)\right) = \frac{1}{1+\mu}\Delta u^{\delta}_{\mathrm{ad}_i}(t)
\end{aligned} \tag{30}
$$

where

$$
\Delta u^{\delta}_{\mathrm{ad}_i}(t) \triangleq u^{\delta}_{\max_i}\operatorname{sat}\left(\frac{u_{\mathrm{ad}_i}(t)}{u^{\delta}_{\max_i}}\right) - u_{\mathrm{ad}_i}(t)
$$

Consequently, if $\Delta u^{\delta}_{\mathrm{ad}_i}(t)$ is uniformly bounded, then the control deficiency $\Delta u_{c_i}(t)$ is inversely proportional to μ; that is, $\Delta u_{c_i}(t) = O(1/\mu)$ uniformly in time.

Substituting Eq. (23) into Eq. (18) yields the following closed-loop system dynamics:

$$
\dot{x}(t) = (A + B\Lambda K_x^{\top}(t))x(t) + B\Lambda(K_r^{\top}(t)r(t) + \Delta u_{\mathrm{ad}}(t)) \tag{31}
$$

where

$$
\Delta u_{\mathrm{ad}}(t) = u_{\max}\operatorname{sat}\left(\frac{u_c(t)}{u_{\max}}\right) - u_{\mathrm{ad}}(t) \tag{32}
$$

defines the deficiency of the linear in parameters adaptive signal $u_{\mathrm{ad}}(t)$.

V. Adaptive Reference Model and Matching Conditions

The system dynamics in Eq. (31) leads to consideration of the following *adaptive* reference model dynamics:

$$
\dot{x}_m(t) = A_m x_m(t) + B_m(r(t) + K_u^{\top}(t)\Delta u_{\mathrm{ad}}(t)) \tag{33}
$$

where $x_m \in \mathbb{R}^n$ is the state of the reference model, A_m is Hurwitz, and $K_u(t)$ is a matrix of adaptive gains to be determined through the stability proof. Comparing Eq. (33) with the system dynamics in Eq. (31), assumptions are formulated that guarantee existence of the adaptive signal with the μ-modification in Eq. (23).

Assumption 1 (Reference model matching conditions):

$$
\begin{aligned}
&\exists K_x^*, K_r^*, K_u^*, B\Lambda(K_x^*)^{\top} = A_m - A, B\Lambda(K_r^*)^{\top} \\
&\quad = B_m, B_m(K_u^*)^{\top} = B\Lambda
\end{aligned} \tag{34}
$$

Remark 2: The true knowledge of the gains K_x^*, K_r^*, and K_u^* is not required, only their existence is assumed. The second and the third matching conditions in Eq. (34) imply that $K_r^* K_u^* = I$.

VI. Error Dynamics and Stability Analysis

Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$
\begin{aligned}
\dot{e}(t) &= A_m e(t) + B\Lambda(\Delta K_x^{\top}(t)x(t) + \Delta K_r^{\top}(t)r(t)) \\
&\quad - B_m \Delta K_u^{\top}(t)\Delta u_{\mathrm{ad}}(t)
\end{aligned} \tag{35}
$$

where $\Delta K_x(t) = K_x(t) - K_x^*$, $\Delta K_r(t) = K_r(t) - K_r^*$, and $\Delta K_u(t) = K_u(t) - K_u^*$ denote the parameter estimation errors. Consider the following adaptation laws:

$$
\begin{aligned}
\dot{K}_x(t) = -\Gamma_x x(t)e^{\top}(t)PB, \qquad \dot{K}_r(t) = -\Gamma_r r(t)e^{\top}(t)PB \\
\dot{K}_u(t) = \Gamma_u \Delta u_{\mathrm{ad}}(t)e^{\top}(t)PB_m
\end{aligned} \tag{36}
$$

where $\Gamma_x = \Gamma_x^{\top} > 0$, $\Gamma_r = \Gamma_r^{\top} > 0$, and $\Gamma_u = \Gamma_u^{\top} > 0$ are corresponding rates of adaptation, and $P = P^{\top} > 0$ solves the algebraic Lyapunov equation

$$
A_m^{\top}P + PA_m = -Q \tag{37}
$$

for some $Q > 0$. Define the following Lyapunov function candidate:

$$
\begin{aligned}
V &= e^{\top}(t)Pe(t) + \operatorname{tr}(\Delta K_x^{\top}(t)\Gamma_x^{-1}\Delta K_x(t)\Lambda) \\
&\quad + \operatorname{tr}(\Delta K_r^{\top}(t)\Gamma_r^{-1}\Delta K_r(t)\Lambda) + \operatorname{tr}(\Delta K_u^{\top}(t)\Gamma_u^{-1}\Delta K_u(t)\Lambda)
\end{aligned} \tag{38}
$$

Its time derivative along the system trajectories (35) and (36) is

$$
\dot{V}(t) = -e^{\top}(t)Qe(t) \leq 0 \tag{39}
$$

Hence the equilibrium of Eqs. (35) and (36) is Lyapunov stable, that is, the signals $e(t)$, $\Delta K_x(t)$, $\Delta K_r(t)$, and $\Delta K_u(t)$ are bounded. Consequently, there exist $\Delta K_x^{\max}$, $\Delta K_r^{\max}$, such that $\|\Delta K_x(t)\| < \Delta K_x^{\max}$, $\|\Delta K_r(t)\| < \Delta K_r^{\max} = \alpha \Delta K_x^{\max}$, $\forall\ t > 0$, where $\alpha = \sqrt{\lambda_{\min}(\Gamma_r)/\lambda_{\min}(\Gamma_x)}$. However, due to the adaptive modification of the reference system, its state may not necessarily remain bounded. This presents an obstacle in applying Barbalat's lemma to prove the closed-loop system stability. To circumvent this situation, we first derive sufficient conditions for the system state to remain bounded. Subsequently, these conditions will allow us to apply Barbalat's lemma and prove closed-loop stability of the system.

To streamline the analysis and to focus on the main ideas in the proof, we set $\mu = 0$, $\delta = 0$, and later formulate a corollary for the case when $\mu \neq 0$, $\delta \neq 0$.

Let P_M and P_m denote the maximum and minimum eigenvalues of the matrix P in Eq. (37). Similarly, let Q_m be the minimum eigenvalue of Q. Further, let $u_{\min} = \min\{u_{\max_1}, \ \ldots \ u_{\max_m}, \}$, $\rho = \frac{P_M}{P_m}$, $\kappa = |Q_m - 2\|PB\Lambda\|\,\|K_x^*\|\,|$, $u_0 \triangleq u_{\min} + \sqrt{m}u_{\max}$.

Theorem 1: For A and B in Eq. (12), $U_{\max}$ in Eq. (14), K_x^*, K_r^* in Eq. (34) and P and Q in Eq. (37), let

$$
r_{\max} < \frac{Q_m - 2u_0\|U_{\max}^{-1}\|_{\infty}\|PB\Lambda\|\|K_x^*\|}{\frac{\kappa\sqrt{\rho}}{u_{\min}}\|K_r^*\|(1 + u_0\|U_{\max}^{-1}\|_{\infty})} \tag{40}
$$

If the system initial condition and the initial value of the Lyapunov function in Eq. (38) satisfy

$$
x^{\top}(0)Px(0) < P_m\left[\frac{2\|PB\Lambda\|}{\kappa}u_{\min}\right]^2 \tag{41}
$$

$$
\begin{aligned}
\sqrt{V(0)} &< \sqrt{\frac{\lambda_{\max}(\Lambda)}{\lambda_{\max}(\Gamma_x)}} \\
&\quad \times \frac{Q_m - 2u_0\|U_{\max}^{-1}\|_{\infty}\|PB\Lambda\|\,\|K_x^*\| - \kappa\sqrt{\rho}\frac{r_{\max}}{u_{\min}}\|K_r^*\|(1 + u_0\|U_{\max}^{-1}\|_{\infty})}{(1 + u_0\|U_{\max}^{-1}\|_{\infty})(2\|PB\Lambda\| + \alpha\kappa\sqrt{\rho}\frac{r_{\max}}{u_{\min}})}
\end{aligned} \tag{42}
$$

where $\lambda_{\max}(\Lambda)$, $\lambda_{\max}(\Gamma_x)$ denote the maximum eigenvalues of Λ and Γ_x correspondingly, then

1) the adaptive system in Eqs. (35) and (36), has bounded solutions $\forall\ r(t)$, $\|r(t)\| \leq r_{\max}$;

2) the tracking error $e(t)$ goes to zero asymptotically, and the system states remain in a compact set:

$$x^\top(t)Px(t) < P_m\left[\frac{2\|PB\Lambda\|}{\kappa}u_{\min}\right]^2, \qquad \forall\ t > 0 \tag{43}$$

Proof: If $\Delta u_{ad}(t) = 0$, then the adaptive reference model dynamics in Eq. (33) reduces to the one in Eq. (22), leading to the following form of the error dynamics in Eq. (35):

$$\dot{e}(t) = A_m e(t) + B\Lambda(\Delta K_x^\top(t)x(t) + \Delta K_r^\top(t)r(t)) \tag{44}$$

Because Eq. (22) defines a stable reference model, then $x_m^*(t)$ is bounded, which together with Eq. (39), and using Barbalat's lemma, leads to asymptotic convergence of the tracking error $e(t)$ to the origin.

If $\Delta u_{ad}(t) \neq 0$, then to prove asymptotic convergence of the tracking error to the origin, one needs to show additionally boundedness of at least one of the two states: either $x_m(t)$ or $x(t)$. To this end, suppose that A is a Hurwitz matrix and consider the following Lyapunov function candidate for the system dynamics:

$$W(x) = x^\top(t)P_A x(t) \tag{45}$$

where $P_A = P_A^\top > 0$ solves the algebraic Lyapunov equation

$$A^\top P_A + P_A A = -Q_A$$

for some positive definite $Q_A > 0$. Because $\Delta u(t) \neq 0$, then $\|u(t)\| \leq u_{\max}\sqrt{m}$, and the system dynamics in Eq. (12) become

$$\dot{x}(t) = Ax(t) + B\Lambda u(t) \tag{46}$$

Consequently

$$\begin{aligned}\dot{W}(x(t)) &= -x^\top(t)Q_A x(t) + 2x^\top(t)P_A B\Lambda u(t)\\ &\leq -(Q_A)_m\|x(t)\|^2 + 2\|x(t)\|\,\|P_A B\Lambda\|u_{\max}\sqrt{m}\end{aligned} \tag{47}$$

For open-loop stable systems it immediately implies that $\dot{W} < 0$ if $\|x\| > 2u_{\max}\sqrt{m}\|P_A B\Lambda\|/(Q_A)_m$, where $(Q_A)_m$ is the minimum eigenvalue of Q_A. Therefore, the system states remain bounded, which leads to boundedness of $\dot{e}(t)$. Therefore $\ddot{V}(t)$ is bounded, and the application of Barbalat's lemma ensures global asymptotic stability of the error dynamics in Eq. (35).

For unstable systems, that is, when A is not Hurwitz, write the system dynamics in the following form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\Lambda(K_x^*)^\top x(t) - B\Lambda(K_x^*)^\top x(t) + B\Lambda u(t)\\ &= A_m x(t) - B\Lambda(K_x^*)^\top x(t) + B\Lambda u(t)\end{aligned} \tag{48}$$

and consider the following Lyapunov function candidate:

$$W = x^\top(t)Px(t) \tag{49}$$

where $P = P^\top > 0$ solves the algebraic Lyapunov equation (37) for some positive definite $Q > 0$. Then

$$\begin{aligned}\dot{W}(x(t)) &= -x^\top(t)Qx(t) - 2x^\top(t)PB\Lambda(K_x^*)^\top x(t)\\ &\quad + 2x^\top(t)PB\Lambda u(t)\end{aligned} \tag{50}$$

Notice that $u_{\min} \leq \|u(t)\| \leq \sqrt{m}u_{\max}$. Consider the following two possibilities:

1) $x^\top(t)PB\Lambda u(t) < -\|x(t)\|\,\|PB\Lambda\|u_{\min}$.
2) $x^\top(t)PB\Lambda u(t) \geq -\|x(t)\|\,\|PB\Lambda\|u_{\min}$.

If $x^\top(t)PB\Lambda u(t) < -\|x(t)\|\,\|PB\Lambda\|u_{\min}$, then it follows from Eq. (50) that

$$\begin{aligned}\dot{W}(x(t)) &\leq -Q_m\|x(t)\|^2 + 2\|PB\Lambda\|\|K_x^*\|\|x(t)\|^2\\ &\quad - 2\|x(t)\|\|PB\Lambda\|u_{\min} \leq |Q_m - 2\|PB\Lambda\|\,\|K_x^*\|\,|\,\|x(t)\|^2\\ &\quad - 2u_{\min}\|PB\Lambda\|\,\|x(t)\|\end{aligned} \tag{51}$$

Therefore $\dot{W} < 0$ if

$$x \in \Omega_1 \triangleq \left\{x \mid \|x\| < 2\|PB\Lambda\|\frac{u_{\min}}{\kappa}\right\} \tag{52}$$

Consider the largest set $\mathcal{B}_1$, enclosed in Ω_1, whose boundary forms a level set of the function $W(x(t))$:

$$\mathcal{B}_1 = \left\{x \mid W(x) \leq P_m\left[\frac{2\|PB\Lambda\|u_{\min}}{\kappa}\right]^2\right\} \tag{53}$$

It is obvious that for all initial conditions of $x(t)$ from the set $\mathcal{B}_1$ we have $\dot{W}(x(t)) < 0$, implying that the system states remain bounded.

If $x^\top(t)PB\Lambda u(t) \geq -\|x(t)\|\,\|PB\Lambda\|u_{\min}$, then using Eq. (16) along with Eq. (26) for $\mu = 0$, we have

$$\begin{aligned}&x^\top(t)PB\Lambda\left[\frac{K_x^\top(t)x(t) + K_r^\top(t)r(t)}{\|U_{\max}^{-1}u_c(t)\|_\infty} + \bar{u}(t)\right]\\ &\quad + \|x(t)\|\,\|PB\Lambda\|u_{\min} \geq 0\end{aligned}$$

or

$$\begin{aligned}&2x^\top(t)PB\Lambda(\Delta K_x^\top(t)x(t) + K_r^\top(t)r(t))\\ &\quad + 2x^\top(t)PB\Lambda\bar{u}(t)\|U_{\max}^{-1}u_c(t)\|_\infty\\ &\quad + 2\|U_{\max}^{-1}u_c(t)\|_\infty\|x(t)\|\,\|PB\Lambda\|u_{\min}\\ &\quad \geq -2x^\top(t)PB\Lambda(K_x^*)^\top x(t)\end{aligned}$$

Therefore, using Eq. (17), and recalling the notation $u_0 \triangleq u_{\min} + \sqrt{m}u_{\max}$, we can upperbound the derivative of the Lyapunov function candidate as follows:

$$\begin{aligned}\dot{W}(x(t)) &= -x^\top(t)Qx(t) - 2x^\top(t)PB\Lambda(K_x^*)^\top x(t)\\ &\quad + 2x^\top(t)PB\Lambda u(t) \leq -Q_m\|x(t)\|^2\\ &\quad + 2\|x(t)\|\,\|PB\Lambda\|(\Delta K_x^{\max}\|x(t)\| + (\Delta K_r^{\max} + \|K_r^*\|)r_{\max})\\ &\quad + 2\|U_{\max}^{-1}\|_\infty\|u_c(t)\|_\infty\|x(t)\|\,\|PB\Lambda\|u_0\end{aligned} \tag{54}$$

Because $\|u_c(t)\|_\infty \leq \|u_c(t)\|$, we can rewrite the above as

$$\begin{aligned}\dot{W}(x(t)) &\leq -Q_m\|x(t)\|^2 + 2\|x(t)\|\,\|PB\Lambda\|(\Delta K_x^{\max}\|x(t)\|\\ &\quad + (\Delta K_r^{\max} + \|K_r^*\|)r_{\max})\\ &\quad + 2\|U_{\max}^{-1}\|_\infty\|u_c(t)\|\,\|x(t)\|\,\|PB\Lambda\|u_0\end{aligned} \tag{55}$$

Substituting for $u_c(t)$ from Eq. (26) and setting $\mu = 0$, we get

$$\begin{aligned}\dot{W}(x(t)) &\leq -Q_m\|x(t)\|^2 + 2\|x(t)\|\,\|PB\Lambda\|(\Delta K_x^{\max}\|x(t)\|\\ &\quad + (\Delta K_r^{\max} + \|K_r^*\|)r_{\max})\\ &\quad + 2\|U_{\max}^{-1}\|_\infty\|x(t)\|\,\|PB\Lambda\|((\Delta K_x^{\max} + \|K_x^*\|)\|x(t)\|\\ &\quad + (\Delta K_r^{\max} + \|K_r^*\|)r_{\max})u_0\end{aligned}$$

Further, grouping the terms, one gets

$$\begin{aligned}\dot{W}(x) &\leq -(Q_m - 2\|PB\Lambda\|\Delta K_x^{\max} - 2u_0\|U_{\max}^{-1}\|_\infty\|PB\Lambda\|(\Delta K_x^{\max}\\ &\quad + \|K_x^*\|))\|x\|^2 + 2\|x\|\,\|PB\Lambda\|(\Delta K_r^{\max}\\ &\quad + \|K_r^*\|)(1 + u_0\|U_{\max}^{-1}\|_\infty)r_{\max}\end{aligned} \tag{56}$$

Notice that since $V(e(t), \Delta K_x(t), \Delta K_r(t), \Delta K_u(t))$ is radially unbounded, and its derivative is negative semidefinite, then the maximal values of all errors, including $\Delta K_x^{\max}, \Delta K_r^{\max}$, do not exceed the level set value of the Lyapunov function $V = V_0 = V(0)$. Therefore using the assumed inequality in Eq. (42) yields

$$\Delta K_x^{\max} < \frac{Q_m - 2u_0\|U_{\max}^{-1}\|_\infty\|PB\Lambda\|\|K_x^*\| - \kappa\sqrt{\rho\frac{r_{\max}}{u_{\min}}}\|K_r^*\|(1+u_0\|U_{\max}^{-1}\|_\infty)}{(1+u_0\|U_{\max}^{-1}\|_\infty)(2\|PB\Lambda\| + \alpha\kappa\sqrt{\rho\frac{r_{\max}}{u_{\min}}})} \tag{57}$$

This in turn guarantees that $Q_m - 2\|PB\Lambda\|\Delta K_x^{\max} - 2u_0\|U_{\max}^{-1}\|_\infty \|PB\Lambda\|(\Delta K_x^{\max} + \|K_x^*\|) > 0$. Consequently, it follows from Eq. (56) that $\dot{W}(x(t)) < 0$ if

$$\|x\| \geq 2\|PB\Lambda\| \frac{(\Delta K_r^{\max} + \|K_r^*\|)(1 + u_0\|U_{\max}^{-1}\|_\infty)r_{\max}}{Q_m - 2\|PB\Lambda\|\Delta K_x^{\max} - 2u_0\|U_{\max}^{-1}\|_\infty\|PB\Lambda\|(\Delta K_x^{\max} + \|K_x^*\|)} \triangleq x_{\min} \tag{58}$$

Define the ball

$$\mathcal{B}_2 = \{x \mid \|x\| \leq x_{\min}\}$$

and the smallest set Ω_2 that encloses $\mathcal{B}_2$, the boundary of which is a level set of the Lyapunov function $W(x(t))$:

$$\Omega_2 = \left\{x \mid W(x(t)) \leq P_M x_{\min}^2\right\}$$

By rearranging the terms in Eq. (57) one arrives at

$$\sqrt{P_M}\frac{x_{\min}}{2\|PB\Lambda\|} < \sqrt{P_m}\frac{u_{\min}}{\kappa}$$

and consequently implies that $\Omega_2 \subset \Omega_1$, implying that there exists at least one level set of Lyapunov function in the set $\Omega_1 \setminus \Omega_2 \neq \emptyset$. Thus our analysis of the closed-loop system dynamics reveals that when $\Delta u(t) \neq 0$, there always exists a *nonempty* annulus region such that $\dot{W}(x(t)) < 0$ holds $\forall\ x$ from that region. In other words, asymptotic convergence of the tracking error to zero and boundedness of all signals are guaranteed as long as the system initial conditions satisfy Eq. (41) and the initial parameter errors comply with Eq. (42).

Remark 3: Inequality in Eq. (40) ensures that the resulting numerator in Eq. (42) is positive.

Remark 4: Theorem 1 implies that if the initial conditions of the state and the parameter errors lie within certain bounds, then the adaptive system will have bounded solutions. The local nature of the result for *unstable* systems is due to the static actuator model constraints (14) imposed on the control input. For open-loop stable systems the results are global.

Corollary 1: We note that the presence of $\mu \neq 0$ and $\delta \neq 0$ in Eq. (29) leads to reduced control activity within the layer $u_{\max_i}^{\delta} \leq u_{c_i} \leq u_{\max_i}$, by modifying its slope for smoother performance. In this case, the sufficient conditions of Theorem 1 take a more conservative form:

$$\sqrt{V(0)} < \frac{Q_m - 2u_0\|U_{\max}^{-1}\|_\infty\|PB\Lambda\|\ \|K_x^*\| - \kappa\sqrt{\rho}\frac{r_{\max}\|K_r^*\| + \mu\tilde{u}_{\max}}{u_{\min}}(1 + u_0\|U_{\max}^{-1}\|_\infty)}{(1 + u_0\|U_{\max}^{-1}\|_\infty)(2\|PB\Lambda\| + \alpha\kappa\sqrt{\rho}\frac{r_{\max} + \mu\tilde{u}_{\max}}{u_{\min}})} \tag{59}$$

where $\tilde{u}_{\max} \triangleq ||\tilde{U}_{\max}||$, and

$$r_{\max}\|K_r^*\| + \mu\tilde{u}_{\max} < \frac{Q_m - 2u_0\|U_{\max}^{-1}\|_\infty\|PB\Lambda\|\ \|K_x^*\|}{\frac{\kappa\sqrt{\rho}}{u_{\min}}(1 + u_0\|U_{\max}^{-1}\|_\infty)} \tag{60}$$

This restriction on $r_{\max}$ limits also the choice of μ. Notice that the domain of attraction for the state vector does not change, because it is independent of the control design and depends only on $u_{\max}$ and the system parameters. Selection of $\mu \neq 0$ and $\delta \neq 0$ can prevent saturation if needed, as demonstrated in [11].

VII. Augmentation Based Design

The adaptive control architecture presented in Sec. IV can be equivalently rewritten as an augmentation of a baseline/nominal controller. The motivation for the latter comes from the fact that in most real-life applications, often there is a need to augment a preexisting baseline controller rather than to replace it. In this case, the reference model can be introduced in a very natural way. Specifically, the reference dynamics is chosen to represent the assumed (no uncertainties) system dynamics operating under the baseline controller. Then the main goal of the adaptive augmentation is to maintain the nominal closed-loop system behavior in the presence of parametric uncertainties and control saturation. Formalizing these notions, the total linear-in-parameters adaptive control signal is defined as

$$u_{\text{ad}}(t) = \left(K_{x_{\text{nom}}}^\top + \hat{K}_x^\top(t)\right)x(t) + \left(K_{r_{\text{nom}}}^\top + \hat{K}_r^\top(t)\right)r(t) \tag{61}$$

where $K_{x_{\text{nom}}}, K_{r_{\text{nom}}}$ are the baseline feedback and feedforward design gains, while $\hat{K}_x(t)$, $\hat{K}_r(t)$ are the incremental adaptive gains. Comparing this to the definition in Eq. (24), it is straightforward to notice that

$$K_x(t) = K_{x_{\text{nom}}} + \hat{K}_x(t), \qquad K_r(t) = K_{r_{\text{nom}}} + \hat{K}_r^\top(t)$$

while the adaptive laws for $\hat{K}_x(t)$, $\hat{K}_r(t)$ are the same as for $K_x(t)$, $K_r(t)$, since the nominal design gains are constant. In the absence of the system uncertainties, that is, when $\Lambda = \mathbb{I}$ and $A = A_{\text{nom}}$, the feedback and feedforward gains $K_{x_{\text{nom}}}, K_{r_{\text{nom}}}$ are calculated such that $A_m = A_{\text{nom}} + BK_{x_{\text{nom}}}^\top$ is Hurwitz, while $K_{r_{\text{nom}}}^\top = -(CA_m^{-1}B)^{-1}$ achieves unity low frequency (DC) gain for the closed-loop system. Thus, the nominal reference system is defined as

$$\dot{x}_m^*(t) = A_m x_m^*(t) + BK_{r_{\text{nom}}}^\top r(t) = A_m x_m(t) + B_m r(t)$$

whereas the modified reference system is

$$\dot{x}_m(t) = A_m x_m(t) + B_m(r(t) + K_u(t)\Delta u_{\text{ad}}(t))$$

It is important to see that with the structure in Eq. (61), the adaptive gains $\hat{K}_x(t)$ and $\hat{K}_r(t)$ must be initialized at zero in Eq. (36), while $K_u(0) = -(CA_m^{-1}B)^{-1}$. Such initialization for $K_u(0)$ is consistent with the comment in Remark 2.

VIII. Simulation Example: Roll/Yaw Adaptive Autopilot Design for an F-16 Aircraft

In this section, the proposed control design methodology is demonstrated using the roll/yaw dynamics of an F-16 aircraft. The open-loop aircraft data are calculated at the nominal flight conditions shown in Table 3.4.-3, [12]. The nominal lateral-directional model of the aircraft has the following matrices:

$$A_{\text{nom}} = \begin{bmatrix} -0.322 & 0.064 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0037 \\ -30.6492 & 0 & -3.6784 & 0.6646 \\ 8.5396 & 0 & -0.0254 & -0.4764 \end{bmatrix}$$
$$B_{\text{nom}} = \begin{bmatrix} 0.0003 & 0.0008 \\ 0 & 0 \\ -0.7333 & 0.01315 \\ -0.0319 & -0.0620 \end{bmatrix} \tag{62}$$

for the dynamics of the states

$$x = \begin{bmatrix} \beta & \phi & p & r \end{bmatrix}^\top$$

where β is the aircraft angle of sideslip (AOS), ϕ is the bank angle, p is the body roll rate, and r is the body yaw rate. The commanded control input

$$u = \begin{bmatrix} \delta_a & \delta_r \end{bmatrix}^\top$$

consists of the commanded aileron and the rudder deflections. Aileron and rudder actuator position limits are

$$\begin{matrix} \delta_{a_{\max}} = 25^\circ \\ \delta_{r_{\max}} = 30^\circ \end{matrix} \qquad \Rightarrow U_{\max} = \frac{1}{57.3}\begin{bmatrix} 25 & 0 \\ 0 & 35 \end{bmatrix} \tag{63}$$

The two controlled outputs of interest are the aircraft bank angle ϕ and AOS β:

$$y = \begin{bmatrix} \phi \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x = Cx \tag{64}$$

In the model, all angles (including controls) are measured in radians, and all angular rates are in radians per second. For this system, the open-loop eigenvalues λ_i^{ol} along with their corresponding damping ratios ζ_i^{ol} and the natural frequencies ω_i^{ol} are

$$\begin{cases} \lambda_1^{\text{ol}} = -0.4226 + 3.0634i \\ \lambda_2^{\text{ol}} = -0.4226 - 3.0634i \\ \lambda_3^{\text{ol}} = -3.6152 \\ \lambda_4^{\text{ol}} = -0.0163 \end{cases} \Rightarrow \begin{cases} \zeta_1^{\text{ol}} = 0.137, & \omega_1^{\text{ol}} = 3.0900 \\ \zeta_2^{\text{ol}} = 0.137, & \omega_2^{\text{ol}} = 3.0900 \\ \zeta_3^{\text{ol}} = 1.000, & \omega_3^{\text{ol}} = 3.6152 \\ \zeta_4^{\text{ol}} = 1.000, & \omega_4^{\text{ol}} = 0.0163 \end{cases} \tag{65}$$

The modal characteristics (65) indicate that the aircraft roll/yaw dynamics are open loop stable. However, the damping ratio of the aircraft dutch-roll model is low. The adaptive controller is designed to augment the baseline controller, as discussed in Sec. VII.

To improve the roll/yaw modes, baseline control is performed using the linear quadratic regulator (LQR) method [12]. The state and the control weighting matrices are chosen as

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \tag{66}$$

The optimal nominal LQR feedback gains are calculated to be

$$K_{x_{\text{LQR}}}^\top = \begin{bmatrix} -11.0538 & 9.5717 & 2.0290 & 6.0872 \\ -2.4319 & 0.3745 & 0.4498 & 27.0780 \end{bmatrix}$$

The feedforward gains are set to

$$K_{r_{\text{LQR}}}^\top = \begin{bmatrix} -9.9882 & -2.592 \\ -3.2316 & 155.4926 \end{bmatrix}$$

These gains are used to form the nominal controller as defined in Eq. (61). In particular, we set $K_{x_{\text{nom}}} = K_{x_{\text{LQR}}}$ and $K_{r_{\text{nom}}} = K_{r_{\text{LQR}}}$. The resulting closed-loop system eigenvalues, damping ratios, and the natural frequencies become

$$\begin{cases} \lambda_1^{\text{cl}} = -1.2418 + 2.9689i \\ \lambda_2^{\text{cl}} = -1.2418 - 2.9689i \\ \lambda_3^{\text{cl}} = -2.6501 + 0.6699i \\ \lambda_4^{\text{cl}} = -2.6501 - 0.6699i \end{cases} \Rightarrow \begin{cases} \zeta_1^{\text{cl}} = 0.386, & \omega_1^{\text{cl}} = 3.22 \\ \zeta_2^{\text{cl}} = 0.386, & \omega_2^{\text{cl}} = 3.22 \\ \zeta_3^{\text{cl}} = 0.970, & \omega_3^{\text{cl}} = 2.73 \\ \zeta_4^{\text{cl}} = 0.970, & \omega_4^{\text{cl}} = 2.73 \end{cases} \tag{67}$$

It can be verified that without the actuator constraints, the baseline controller in the form

$$u_{\text{nom}}(t) = K_{x_{\text{LQR}}}^\top x(t) + K_{r_{\text{LQR}}}^\top r(t)$$

achieves adequate roll and AOS step-input tracking characteristics with minimal cross-coupling effects in the roll and yaw channels.

Next, uncertainties are introduced into the system so that

$$A = A_{\text{nom}} + B_{\text{nom}} \Lambda F^\top$$

where the uncertainty matrix F is

$$F^\top = \begin{bmatrix} 16.4763 & 0.07 & -7.1805 & -2.8653 \\ -182.2362 & 0.0 & 4.8993 & -12.6566 \end{bmatrix}$$

while the uncertainties in control effectiveness are set to

$$\Lambda = \begin{bmatrix} 0.95 & 0.00 \\ 0.00 & 0.80 \end{bmatrix}$$

which corresponds to 5% reduction in the roll control (aileron) effectiveness and 20% reduction in the yaw control (rudder) effectiveness, respectively. With these uncertainties, the open-loop system becomes unstable. Its eigenvalues are calculated to be

$$\begin{cases} \lambda_1^{\text{ol}} = 0.1384 + 4.3268i \\ \lambda_2^{\text{ol}} = 0.1384 - 4.3268i \\ \lambda_3^{\text{ol}} = 1.4540 \\ \lambda_4^{\text{ol}} = -0.0874 \end{cases}$$

These uncertainties were found to represent the worst-case scenario in the sense that in their presence the total control was still able to provide adequate tracking, while the same uncertainties lead to instability under the nominal controller. The control objective is to control an unstable airframe in the presence of system uncertainties and control saturation, while tracking roll and sideslip commands. Virtual control limits are defined using $\delta = 0.2$:

$$\tilde{U}_{\max} = \left(\frac{1-\delta}{57.3}\right)\begin{pmatrix} 25 & 0 \\ 0 & 35 \end{pmatrix}$$

Rates of adaptation in Eq. (36) are chosen as

$$\Gamma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}, \qquad \Gamma_r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_u = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.0001 \end{pmatrix}$$

The Lyapunov equation in (37) is solved with

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{pmatrix}$$

The total control is defined as in Eq. (61). Adaptive laws for the gains $\hat{K}_x(t)$ and $\hat{K}_r(t)$ are defined as in Eq. (36), where $K_x(t) = \hat{K}_x(t)$ and $K_r(t) = \hat{K}_r(t)$ represent the adaptive augmentation gains. The adaptive gain K_u is initialized to satisfy the matching conditions (34), while using the nominal controller feedforward gain $K_{r_{\text{nom}}} = K_{r_{\text{LQR}}}$.

Case 1: Nominal Control. First, the adaptive control, the μ-mod and the actuator constraints were all turned off and the system baseline tracking performance was evaluated in the presence of uncertainties. The control goal was to track roll commands while maintaining zero AOS. The two commands represented a typical coordinated flight simulation scenario. Figure 3 shows the results. The data clearly indicate a significant performance degradation due to the system uncertainties.

Next, the control saturation was turned on but the adaptation and the μ-modification were turned off. The system was simulated again

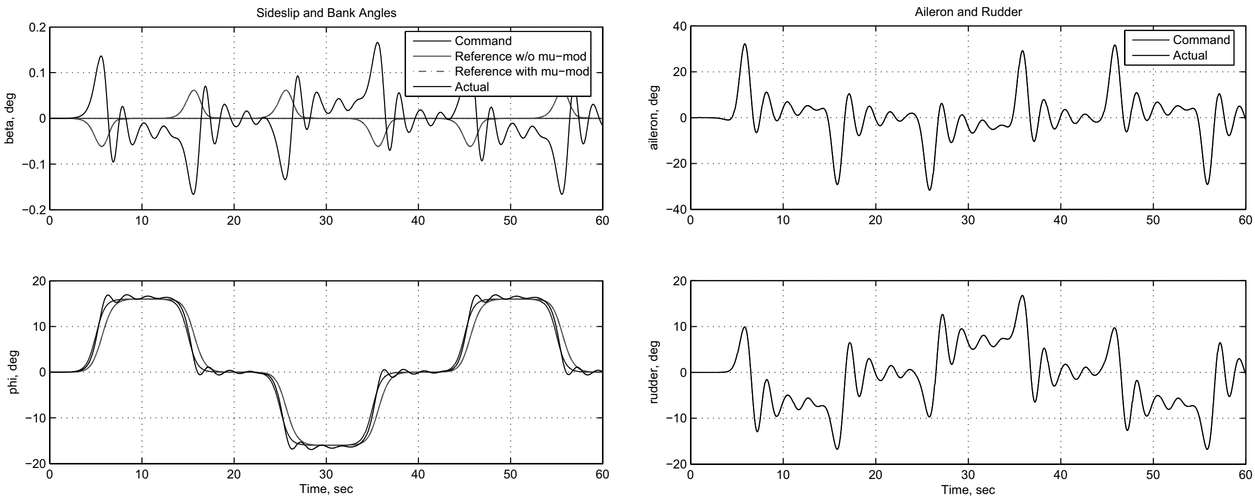

Fig. 3 Baseline control: Sideslip and bank angle tracking (left); aileron and rudder deflections (right).

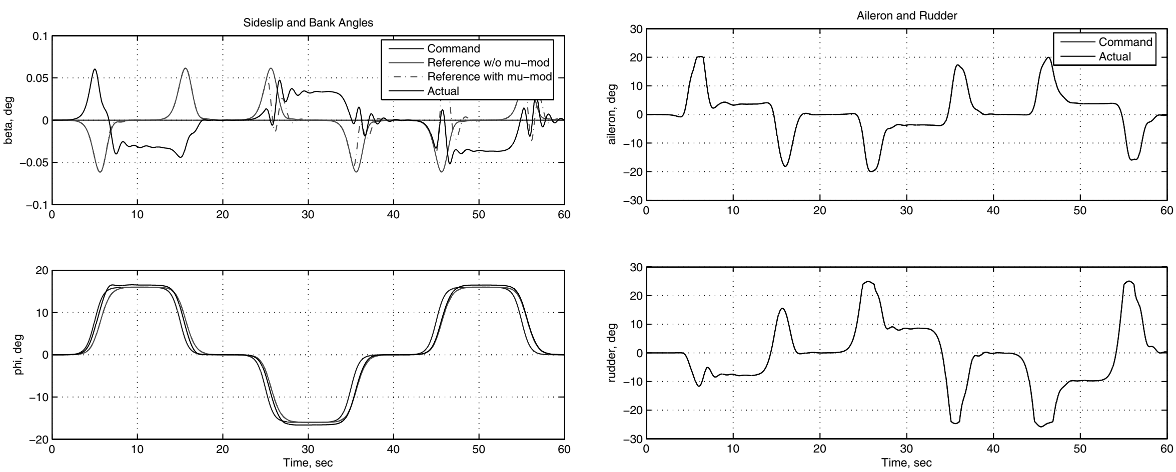

Fig. 4 Sideslip and bank angle tracking (left); aileron and rudder deflections (right).

using the baseline control only. It was immediately observed that the actuator saturation and the uncertainties quickly drove the closed-loop system to instability and the aircraft departed.

Case 2: Adaptive Augmentation. Finally, the system tracking performance was tested with all of the control components turned on and using $\mu = 10$. The tracking performance and control efforts data are shown in Fig. 4. As seen from the figure, the aircraft tracks bank angle command with a minimum variation in the AOS. Moreover, the adaptive system quickly "learns" the uncertainties and shows significant improvement when the roll command is repeated.

Additionally, the system tracking performance was also verified for an external command that represented a sum of sinusoids. The results are shown in Fig. 5. The data clearly demonstrate the two main benefits of the μ-modification. First, the control saturation is prevented at all times. Second, the reference model is modified and, as a result, the system tracking performance is recovered and

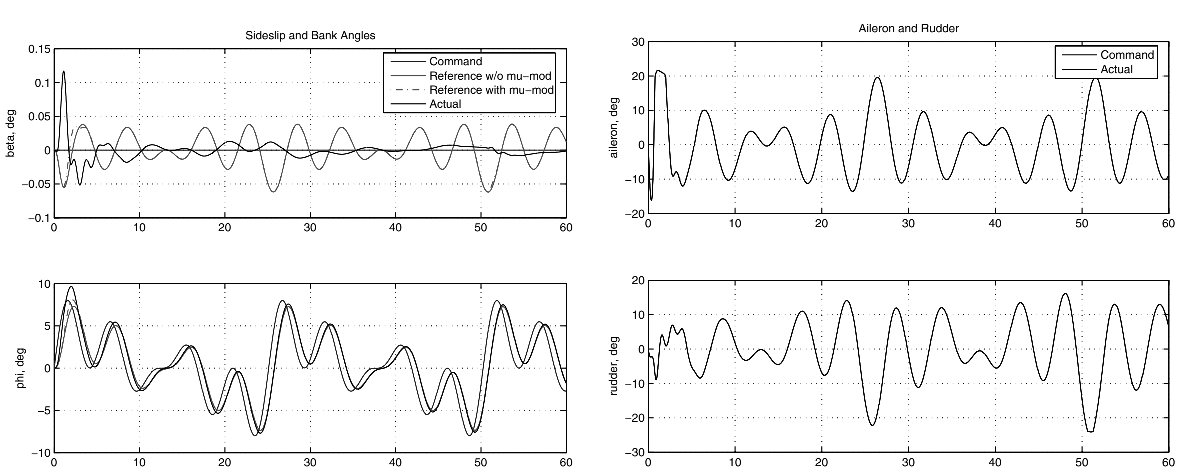

Fig. 5 Sideslip and bank angle tracking (left); aileron and rudder deflections (right).

maintained, while the system operates under the actuator constraints, in the presence of control failures, and with matched uncertainties.

IX. Conclusions

A direct adaptive model reference control design methodology is developed for uncertain multi-input linear systems in the presence of position limited actuators. The method has the option of preventing control saturation if needed. For unstable systems, an explicit domain of attraction is constructed, while for stable systems the method is shown to ensure global asymptotic stability. The methodology is demonstrated using F-16 aircraft linear roll/yaw dynamics. Future work will address the extension of this methodology to rate saturation.

Acknowledgment

This material is based upon work supported by the United States Air Force under Contracts No. FA9550-04-C-0047 and No. FA9550-05-1-0157.

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